Infrared decimation renormalization-group calculations for two-dimensional test-field turbulence

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We perform a decimation of modes starting from the infrared end of wave numbers on the randomly stirred test-field dynamics augmented by linear (Rayleigh) drag terms to model turbulence in two dimensions. A renormalization-group scheme shows relevant corrections to the drag coefficients and the existence of ultraviolet attractive fixed points, facilitating calculations of the universal numbers in both the energy and the enstrophy regimes. Marginal behavior in the enstrophy range yields logarithmic renormalization. We make a detailed comparison of the renormalization-group results with the numerical and analytical results following from Kraichnan's test-field closure.

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I. INTRODUCTION

The most intriguing feature of two-dimensional strong turbulence is the existence of two different conservative cascades—an inverse cascade of energy, from small to large scales, with the Kolmogorov spectrum [1–6]

$$E(k) = C\bar{\varepsilon}^{2/3}k^{-5/3} \tag{1}$$

and a direct cascade of enstrophy, from large to small scales, with the Kraichnan-Batchelor spectrum [3–7]

$$E(k) = C'\bar{\chi}^{2/3}k^{-3}(\ln k/k_0)^{-1/3}. \tag{2}$$

This situation is grossly different from that in three dimensions, where only a direct cascade (with the Kolmogorov spectrum) exists, that due to conservative transport of energy [1,2].

Two-dimensional turbulence has been of interest among various researchers and comprehensive reviews could be found in Refs. [8–10]. Various recent physical experiments [11–13] have favored the existence of both these regimes with spectra very close to $k^{-5/3}$ and k^{-3}, respectively. However, the $k^{-5/3}$ regime was not observed in an early numerical simulation [14] due to emergence of coherent vortices. Another simulation [15] showed the existence of this regime until a Bose-Einstein condensation takes place due to finite size of the system. It has been argued [16] that the existence of this regime also depends on a proper (linear) drag representation operating at large scales. A linear (Rayleigh) drag modeling [17] has been experimentally seen to be appropriate for modeling the frictional coupling of the 2D fluid with its 3D environment. A numerical simulation with a linear drag modeling [52] has in fact supported the existence of the $k^{-5/3}$ regime. The numerical support for the k^{-3} enstrophy range was obtained in an early numerical simulation [18]. A recent numerical simulation [53] also yields the k^{-3} scaling and gives a convincing evidence for the logarithmic renormalization.

It was Kraichnan [19,20] who formulated the first "microscopic" theory of (three-dimensional) turbulence, namely the Direct-Interaction approximation (akin to Dyson-Schwinger equations in Quantum Field Theory [21,22]), based on the underlying Navier-Stokes dynamics of fluid-motion. However, the Direct-Interaction Approximation (DIA) was unable to reproduce the Kolmogorov $k^{-5/3}$ spectrum due to a divergence in the response-integral coming from low wave numbers [23]. Kraichnan realized this departure of the DIA to be associated with a sweeping of the smaller scales by the larger ones, leading to a spurious non-Kolmogorov ($k^{-3/2}$) spectrum. Motivated by such an idea, Kraichnan reformulated the theory in a Lagrangian frame-work [24], which eliminated the sweeping in a systematic fashion, thereby yielding the Kolmogorov spectrum. However, the Lagrangian framework is formidably cumbersome, and Kraichnan [25] considered the (Eulerian) problem of advection of the solenoidal and compressive parts of a "test-field." On removing the self-advection terms in the respective dynamical equations, and giving a similar treatment like the DIA together with implementing a Markovianized scheme, he obtained a theory capable of reproducing not only the Kolmogorov spectrum, but also the Kraichnan-Batchelor spectrum including the logarithmic correction. Furthermore, he calculated [6] the Kolmogorov constant numerically, yielding $C=1.4$ (in three dimensions) and $C=6.69$ (in two dimensions). Marginal behavior of the response integral in the enstrophy cascade allowed *analytic* computation, yielding the logarithmic correction and $C'=2.262$.

The conventional Wilsonian RG scheme [26–28], which recursively decimates modes starting from the ultraviolet (small-scale) end, has been implemented for dynamical systems, for example criticial dynamics of ferromagnetic systems by Ma and Mazenko [29] and the case of a Navier-Stokes fluid near thermal equilibrium by Forster *et al.* [30]. DeDominicis and Martin [31] proposed a randomly stirred model and obtained the Kolmogorov spectrum within a field-theoretic RG scheme. The Wilsonian RG scheme was used

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by Yakhot and Orszag [32] for the randomly stirred model to calculate the Komogorov constant and various other universal numbers ($C=1.6057$, in three dimensions, see also [33,34]). The field-theoretic RG [36–38] was applied in the two-dimensional problem [39] of randomly stirred Navier-Stokes dynamics yielding $C=6.447$ (8.123) and $C'=1.59$ (2.423) [40,41], see also [35]. The field-theoretic RG has been used for various cases recently [42]. However, the applicability of RG to the problem of turbulence has been regarded with doubt [43]. As noted earlier, the DIA (or any Eulerian closure theory based on the Navier-Stokes dynamics) is liable to be inconsistent with the Kolmogorov phenomenology because of limitation due to divergence coming from the spurious sweeping effects. However, Krachnan's test-field based closure is free of this problem and exact results (numerical and analytical) within this closure scheme are already available due to Kraichnan [6]. Thus, it would be interesting to compare the results of an RG analysis based on Kraichnan's test-field equations with the numerical and analytical results already available within Kraichnan's test-field closure.

We would also like to note that a recent experiment by Rivera and Wu [17] studied the dissipation mechanisms in the two-dimensional turbulence in a driven soap film. Since any two-dimensional fluid interacts with its three-dimensional environment, they modeled this coupling by a linear (Rayleigh) drag term in the Navier-Stokes equation, and found that energy is mostly dissipated due to the film-air frictional interaction whereas enstrophy is predominantly dissipated by viscosity. They also predicted the directions of the two cascades consistent with Kraichnan-Batchelor phenomenology.

In this paper, we take the test-field equations of Kraichnan and introduce linear drag terms in order to model the drag operating at large scales, as suggested by the above experiment. Starting with these equations, we perform a RG analysis by means of decimation of modes starting with the infrared (IR) end of wave numbers (large scales). It may be noted that this infra-red decimation is the *reverse* of the Wilsonian RG scheme which decimaets modes from the ultraviolet end. We also note that, having chosen the test-field model, the problem of the crossover occurring at $y=3$ which is believed to persist [43,46] in a RG framework based on the Navier-Stokes dynamics, does not occur in our present RG scheme.

The distinctive feature of the present renormalization-group calculation is that the Rayleigh drag coefficients undergo relevant corrections due to the elimination of the infrared scales. This suggests that such a mechanism of linear drag is "in tune" with the nonlinear interactions relevant to the cascade processes. Further, the renormalization-group flow shows the existence of ultraviolet (UV) attractive fixed points in the energy and enstrophy regimes. This facilitates the evaluation of the universal numbers in the two regimes; the above spectra are obtained including the logarithmic renormalization due to a marginal behavior. We make a detailed comparison of the RG results with the numerical and analytical results following from Kraichnan's test-field closure [6].

II. BRIEF REVIEW OF KRAICHNAN'S MODEL

The incompressible fluid motion is governed by the Navier-Stokes equation (NSE) [45]

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu_0 \nabla^2 u_i \tag{3}$$

where $u_i(\mathbf{x},t)$ is the velocity field, $p(\mathbf{x},t)$ is the pressure field divided by density, and ν_0 the kinematic viscosity of the fluid. The pressure field is related to the velocity field through a Poisson equation because of the incompressibility condition $\nabla \cdot \mathbf{u}=0$ coming from the equation of continuity.

Kraichnan took a Fourier transformed version of Eq. (3) for its closure under the Direct Interaction Approximation [19]. However, as stated earlier, the DIA suffered form the difficulty in representing the Kolmogorov regime due to spurious sweeping effects coming from interaction between eddies widely separated in size. This drawback happens to be a feature of any Eulerian theory of turbulence. Kraichnan reformulated the problem in a Lagrangian framework [24] which eliminated the sweeping effects systematically and thereby resulting in consistency with the Kolmogorov phenomenology. This itself could be regarded as a great intellectual achievement in the field of turbulence. However, the Lagrangian formulation happens to be too cumbersome mathematically, and Kraichnan gave a simpler formulation within the Eulerian framework [25,6]. His main objective was to estimate the effects of distortion due to pressure without neglecting the advection term. He considered the model problem of the pressureless advection of a passive vector field $v_i(\mathbf{x},t)$ (the test-field) obeying

$$\frac{\partial v_i}{\partial t} + u_j \frac{\partial v_i}{\partial x_j} = \nu_0 \nabla^2 v_i, \tag{4}$$

where the advecting field $u_i(\mathbf{x},t)$ is purely solenoidal, $\nabla \cdot \mathbf{u} =0$, whereas the test field has both solenoidal ($\mathbf{v}^S$) and compressive ($\mathbf{v}^C$) parts; $\mathbf{v}=\mathbf{v}^S+\mathbf{v}^C$, with $\nabla \cdot \mathbf{v}^S=0$ and $\nabla \times \mathbf{v}^C=0$. Kraichnan considered the Fourier-transformed version of Eq. (4), given by

$$\left(\frac{\partial}{\partial t} + \nu_0 k^2\right) v_i(\mathbf{k},t) = -ik_l \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} u_l(\mathbf{p},t) v_i(\mathbf{q},t), \tag{5}$$

where the summation sign represents a wave-vector integration with the constraint as indicated. From this equation, it is easy to obtain the equations for the solenoidal and compressive parts, namely $v_i^S(\mathbf{k},t)=P_{ij}(\mathbf{k})v_j(\mathbf{k},t)$ and $v_i^C(\mathbf{k},t) =\Pi_{ij}(\mathbf{k})v_j(\mathbf{k},t)$, where $P_{ij}(\mathbf{k})=(\delta_{ij}-k_ik_j/\mathbf{k}^2)$ and $\Pi_{ij}(\mathbf{k}) =k_ik_j/\mathbf{k}^2$. In order to estimate the effects due to pressure, Kraichnan dropped the self-advection terms and retained only the cross-coupling terms, because such dynamics would be absent in the presence of pressure. The resulting dynamical equations become

$$\left(\frac{\partial}{\partial t} + \nu_0 k^2\right) v_i^S(\mathbf{k},t) = -ik_l P_{ij}(\mathbf{k}) \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} u_l(\mathbf{p},t) v_j^C(\mathbf{q},t), \tag{6}$$

$$\left(\frac{\partial}{\partial t}+\nu_0 k^2\right)v_i^C(\mathbf{k},t) = -ik_l\Pi_{ij}(\mathbf{k})\sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} u_l(\mathbf{p},t)v_j^S(\mathbf{q},t). \quad (7)$$

Kraichnan closed these equations with his original direct interaction approximation. In addition, he considered a Markovianized scheme [25,6] where the memory integrals are replaced by terms with no memory. Further, he identified the correlation of the solenoidal part with that of the actual velocity field: $\langle v_i^S(\mathbf{k},t)v_j^S(\mathbf{k}',t')\rangle = \langle u_i(\mathbf{k},t)u_j(\mathbf{k}',t')\rangle = Q(k,t,t')P_{ij}(\mathbf{k})\delta(\mathbf{k}+\mathbf{k}')$. The results for the respective Green functions were

$$\left(\frac{\partial}{\partial t}+\nu_0 k^2\right)G^S(k,t,t') = -\eta^S(k,t)G^S(k,t,t'), \quad (8)$$

$$\left(\frac{\partial}{\partial t}+\nu_0 k^2\right)G^C(k,t,t') = -\eta^C(k,t)G^C(k,t,t'). \quad (9)$$

For stationary turbulence,

$$\eta^S(k) = \tilde{g}^2k^2\sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} b^S(k,q,p)\frac{Q(p)}{\eta^S(k)+\eta^C(q)+\eta^S(p)}, \quad (10)$$

$$\eta^C(k) = 2\tilde{g}^2k^2\sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} b^S(k,q,p)\frac{Q(p)}{\eta^C(k)+\eta^S(q)+\eta^S(p)}, \quad (11)$$

where $b^S(k,q,p)$ is a geometrical factor. Kraichnan introduced the scaling factor $\tilde{g}$ because the model is equally plausible when the characteristic times are scaled; this was fixed by considering equlibrium case where the DIA is expected to be exact, yielding $\tilde{g}=1.064$.

The appearance of three η's in the denominators in Eqs. (10) and (11) in this scheme is an indirect consequence of Markovianization via claiming consistency with energy conservation. Kraichnan took the energy transport equation of the original DIA given by

$$\left(\frac{\partial}{\partial t}+2\nu_0 k^2\right)Q(k,t,t) = 2k^2\sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} b(k,q,p)\frac{Q(p)[Q(q)-Q(k)]}{\eta^S(k)+\eta^S(q)+\eta^S(p)} \quad (12)$$

which is consistent with Kolmogorov scaling. A symmetrized form of Eq. (12) appears in Kraichnan's paper. Equations (10)–(12) constitute a closed set of self-consistent equations. As quoted earlier, Kraichnan solved these equations [6] numerically in the Kolmogorov regime in both three and two dimensions to obtain numerical values for the Kolmogorov constant. In the enstrophy regime in two dimensions these equations are marginal and Kraichnan carried out analytical calculations [6] to obtain a value for the Kraichnan-Batchelor constant C' and the logarithmic correction.

III. PRESENT FORMULATION

In this paper we aim to carry out a renormalization-group analysis for two-dimensional turbulence starting with Kraichnan's test-field model equations, namely, Eqs. (6) and (7). We add linear drag terms (with Rayleigh drag coefficients η_0 and ζ_0, below) in order to model the drag operating at large scales because of frictional interaction of the two-dimensional fluid with its three-dimensional environment, as suggested by the experiment in Ref. [17]. Adding also a random stirring force field $\mathbf{f}(\mathbf{x},t)$, the fully Fourier-transformed equations take the forms

$$(-i\omega+\eta_0+\nu_0k^2)v_i^S(\mathbf{k},\omega) = f_i^S(\mathbf{k},\omega) - i\lambda_0k_lP_{ij}(\mathbf{k})\int\frac{d^2\mathbf{p}d\omega_1}{(2\pi)^3}u_l(\mathbf{p},\omega_1)v_j^C(\mathbf{q},\omega-\omega_1), \quad (13)$$

$$(-i\omega+\zeta_0+\nu_0k^2)v_i^C(\mathbf{k},\omega) = f_i^C(\mathbf{k},\omega) - i\lambda_0k_l\Pi_{ij}(\mathbf{k})\int\frac{d^2\mathbf{p}d\omega_1}{(2\pi)^3}u_l(\mathbf{p},\omega_1)v_j^S(\mathbf{q},\omega-\omega_1) \quad (14)$$

in two dimensions, where $\lambda_0(=1)$ is a formal expansion parameter [47]. An infrared (low wave number) cutoff at a wave number Λ to the wave-vector integrations is assumed corresponding to the "external" scale, characterized by the scale at which drag operates in the case of the energy regime or by the injection scale in the case of the enstrophy regime. We shall assume that the external driving fields have Gaussian white noise statistics, and the solenoidal part has the correlation

$$\langle f_i^S(\mathbf{k}_1,\omega_1)f_j^S(\mathbf{k}_2,\omega_2)\rangle = F^S(k_1)P_{ij}(\mathbf{k}_1)(2\pi)^2\delta^2(\mathbf{k}_1+\mathbf{k}_2)\times(2\pi)\delta(\omega_1+\omega_2) \quad (15)$$

with $F^S(k)=2D_0/k^{4-\epsilon}$ where ϵ is an external parameter.

IV. INFRARED DECIMATION

The usual recursive decimation type RG tretment for dynamical systems consists of eliminating modes starting with the UV end of the spectrum [29,30,32]. In this paper, the RG treatment that we carry out consists of eliminating modes starting with the IR end. Thus we eliminate the modes $\mathbf{U}^<(\mathbf{k},\omega)$ belonging to the band $\Lambda\le k\le\Lambda e^r$, by means of integrating away these modes. (As stated earlier, Λ is a low wave number cutoff.) Since the elimination begins at the IR end, we neglect the viscosity ν_0, being ineffective at large scales. The effect of such elimination on the UV modes, belonging to $\Lambda e^r<k<\infty$, is then reflected through (relevant) corections to the drag coefficients η_0 and ζ_0 because of the self-energy terms occurring in

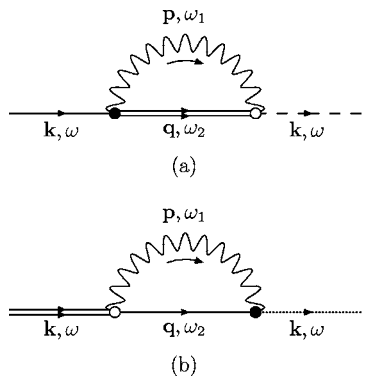

FIG. 1. One-loop Feynman diagrams for (a) self-energy correction to the drag coefficient η_0 and (b) self-energy correction to the drag coefficient ζ_0. The straight single lines represent the response G_0^S, the double lines the response G_0^C, and the wiggly lines the velocity correlation. The filled and open circles represent the vertices coming from Eqs. (16) and (17), respectively. The dashed and dotted lines are the solenoidal and compressive parts of the test-field, respectively. There is a conservation of four-wave-vectors at each vertex so that $\mathbf{p}+\mathbf{q}=\mathbf{k}$ and $\omega_1+\omega_2=\omega$.

$$(-i\omega+\eta_0)v_i^{S>}(\mathbf{k},\omega) = f_i^{S>}(\mathbf{k},\omega) - i\lambda_0 k_l P_{ij}(\mathbf{k}) \int \frac{d^2\mathbf{p}d\omega_1}{(2\pi)^3} u_l^{>}(\mathbf{p},\omega_1) \times v_j^{C>}(\mathbf{q},\omega-\omega_1) - \Sigma_{ij}^S(\mathbf{k},\omega)v_j^{S>}(\mathbf{k},\omega) + \cdots, \tag{16}$$

$$(-i\omega+\zeta_0)v_i^{C>}(\mathbf{k},\omega) = f_i^{C>}(\mathbf{k},\omega) - i\lambda_0 k_l \Pi_{ij}(\mathbf{k}) \int \frac{d^2\mathbf{p}d\omega_1}{(2\pi)^3} u_l^{>}(\mathbf{p},\omega_1) \times v_j^{S>}(\mathbf{q},\omega-\omega_1) - \Sigma_{ij}^C(\mathbf{k},\omega)v_j^{C>}(\mathbf{k},\omega) + \cdots, \tag{17}$$

where the self-energies are expressed by (see Fig. 1) [48]

$$\Sigma_{ik}^S(\mathbf{k},\omega) = \Sigma^S(k,\omega)P_{ik}(\mathbf{k}) = \lambda_0^2 \int \frac{d^2\mathbf{p}d\omega_1}{(2\pi)^3} B_{ik}^S(\mathbf{k},\mathbf{q},\mathbf{p}) Q_0^{<}(\mathbf{p},\omega_1) \times G_0^{C<}(\mathbf{q},\omega-\omega_1), \tag{18}$$

$$\Sigma_{ik}^C(\mathbf{k},\omega) = \Sigma^C(k,\omega)\Pi_{ik}(\mathbf{k}) = \lambda_0^2 \int \frac{d^2\mathbf{p}d\omega_1}{(2\pi)^3} B_{ik}^C(\mathbf{k},\mathbf{q},\mathbf{p}) Q_0^{<}(\mathbf{p},\omega_1) G_0^{S<}(\mathbf{q},\omega-\omega_1), \tag{19}$$

with the p integration in the range $\Lambda \le p \le \Lambda e^r$, and

$$B_{ik}^S(\mathbf{k},\mathbf{q},\mathbf{p}) = k_l P_{ij}(\mathbf{k}) q_m \Pi_{jk}(\mathbf{q}) P_{lm}(\mathbf{p}), \tag{20}$$

$$B_{ik}^C(\mathbf{k},\mathbf{q},\mathbf{p}) = k_l \Pi_{ij}(\mathbf{k}) q_m P_{jk}(\mathbf{q}) P_{lm}(\mathbf{p}), \tag{21}$$

and the bare propagators and the bare correlation are given by

$$G_0^{S<}(k,\omega) = (-i\omega+\eta_0)^{-1} \quad \text{and} \quad G_0^{C<}(k,\omega) = (-i\omega+\zeta_0)^{-1} \tag{22}$$

$$\langle u_i^{<}(\mathbf{k},\omega)u_j^{<}(\mathbf{k}',\omega')\rangle = Q_0^{<}(\mathbf{k},\omega)P_{ij}(\mathbf{k})(2\pi)^2\delta^2(\mathbf{k}+\mathbf{k}') \times (2\pi)\delta(\omega+\omega') \tag{23}$$

as a result of the assumption of isotropy and stationarity.

We note that the coupling constant λ_0 does not undergo any relevant correction due to such elimination process, a consequence of Galelian invariance [30]. Further, the constant D_0 (appearing in the stirring correlation) does not undergo any relevant correction, because of our assumption that the noise correlations scale like $\sim k^{-4+\epsilon}$.

Quite like Kraichnan [25,6], we assume $\langle v_i^S(\mathbf{k},\omega)v_j^S(\mathbf{k}',\omega')\rangle = \langle u_i(\mathbf{k},\omega)u_j(\mathbf{k}',\omega')\rangle$, so that $Q_0^{<}(k,\omega) = |G_0^{S<}(k,\omega)|^2 F^S(k)$. After performing the frequency-convolutions in Eqs. (18) and (19), we obtain

$$\Sigma^S(k,\omega) = \Sigma_{ii}^S(\mathbf{k},\omega) = \lambda_0^2 \int \frac{d^2\mathbf{p}}{(2\pi)^2} B_{ii}^S(\mathbf{k},\mathbf{q},\mathbf{p}) \frac{F^S(p)}{2\eta_0} \frac{1}{-i\omega+\eta_0+\zeta_0}, \tag{24}$$

$$\Sigma^C(k,\omega) = \Sigma_{ii}^C(\mathbf{k},\omega) = \lambda_0^2 \int \frac{d^2\mathbf{p}}{(2\pi)^2} B_{ii}^C(\mathbf{k},\mathbf{q},\mathbf{p}) \frac{F^S(p)}{2\eta_0} \frac{1}{-i\omega+2\eta_0}, \tag{25}$$

with

$$B_{ii}^S(\mathbf{k},\mathbf{q},\mathbf{p}) = B_{ii}^C(\mathbf{k},\mathbf{q},\mathbf{p}) = k^2 \left\{1 - \left(\frac{\mathbf{k}\cdot\mathbf{p}}{kp}\right)^2\right\}\left\{1 - \left(\frac{\mathbf{k}\cdot\mathbf{q}}{kq}\right)^2\right\}. \tag{26}$$

Now we expand the integrands in Eqs. (24) and (25) in the limit $p \ll k$, and perform the angular integrations and subsequently the integration on p from Λ to Λe^r, yielding relevant (k-independent) corrections to the bare drag coefficients as

$$\Sigma^S(\mathrm{k},0) = \frac{3}{16\pi} \frac{\lambda_0^2 D_0}{\eta_0(\eta_0+\zeta_0)} \left(\frac{e^{\epsilon r}-1}{\epsilon}\right) \Lambda^{\epsilon}, \tag{27}$$

$$\Sigma^C(\mathrm{k},0) = \frac{3}{16\pi} \frac{\lambda_0^2 D_0}{2\eta_0^2} \left(\frac{e^{\epsilon r}-1}{\epsilon}\right) \Lambda^{\epsilon}. \tag{28}$$

Consequently, Eqs. (16) and (17) take the forms

$$[-i\omega+\eta_I(r)]v_i^{S>}(\mathbf{k},\omega) = f_i^{S>}(\mathbf{k},\omega) - i\lambda_I(r)k_l P_{ij}(\mathbf{k}) \times \int \frac{d^2\mathbf{p}d\omega_1}{(2\pi)^3} u_l^{>}(\mathbf{p},\omega_1) v_j^{C>}(\mathbf{q},\omega-\omega_1), \tag{29}$$

$$[-i\omega+\zeta_I(r)]v_i^{C>}(\mathbf{k},\omega) = f_i^{C>}(\mathbf{k},\omega) - i\lambda_I(r)k_l\Pi_{ij}(\mathbf{k}) \times\int \frac{\mathrm{d}^2\mathbf{p}\mathrm{d}\omega_1}{(2\pi)^3}u_l^{>}(\mathbf{p},\omega_1)v_j^{S>}(\mathbf{q},\omega-\omega_1), \tag{30}$$

where the corrected (or intermediate) values of the drag-coefficients, resulting from the above elimination of the band $\Lambda < p < \Lambda e^r$, are given by

$$\eta_I(r) = \eta_0 + \Sigma^S(\mathrm{k},0) = \eta_0\left\{1+\frac{3}{16\pi}\frac{g_0}{1+\kappa_0}\frac{e^{\epsilon r}-1}{\epsilon}\right\}, \tag{31}$$

$$\zeta_I(r) = \zeta_0 + \Sigma^C(\mathrm{k},0) = \zeta_0\left\{1+\frac{3}{16\pi}\frac{g_0}{2\kappa_0}\frac{e^{\epsilon r}-1}{\epsilon}\right\}, \tag{32}$$

while

$$\lambda_I(r) = \lambda_0 \quad \text{and } D_I(r) = D_0 \tag{33}$$

as they acquire no corrections. In the above equations, we have defined the bare coupling constants as

$$g_0 = \frac{\lambda_0^2 D_0}{\eta_0^3}\Lambda^{\epsilon} \quad \text{and } \kappa_0 = \frac{\zeta_0}{\eta_0}. \tag{34}$$

V. RESCALING

Now we rescale as

$$\mathbf{k} \to \mathbf{k}' = \mathbf{k}e^{-r} \tag{35}$$

so that the "reduced" range $\Lambda e^r < k < \infty$ is projected on to the "full" range $\Lambda < k' < \infty$. In doing so, we assume that the other quantities are modified to the corresponding primed quantities as follows:

$$\omega' = \omega e^{\phi(r)}, \tag{36}$$

$$\mathbf{U}^{>}(\mathbf{k},\omega) = \psi(r)\mathbf{U}'(\mathbf{k}',\omega'), \tag{37}$$

where $\phi(r)$ and $\psi(r)$ are to be determined.

After this rescaling, we must get the dynamical equations in the same form as the original ones. This condition of RG invariance demands that we get the equations in terms of the primed variables in the following forms:

$$[-i\omega'+\eta(r)]v_i^{S\prime}(\mathbf{k}',\omega') = f_i^{S\prime}(\mathbf{k}',\omega') - i\lambda(r)k_l' P_{ij}(\mathbf{k}') \times\int \frac{\mathrm{d}^2\mathbf{p}'\mathrm{d}\omega_1'}{(2\pi)^3}u_l'(\mathbf{p}',\omega_1')v_j^{C\prime}(\mathbf{q}',\omega'-\omega_1'), \tag{38}$$

$$[-i\omega'+\zeta(r)]v_i^{C\prime}(\mathbf{k}',\omega') = f_i^{C\prime}(\mathbf{k}',\omega') - i\lambda(r)k_l'\Pi_{ij}(\mathbf{k}') \times\int \frac{\mathrm{d}^2\mathbf{p}'\mathrm{d}\omega_1'}{(2\pi)^3}u_l'(\mathbf{p}',\omega_1')v_j^{S\prime}(\mathbf{q}',\omega'-\omega_1'), \tag{39}$$

where $\eta(r)$, $\zeta(r)$, and $\lambda(r)$ are scaled drag coefficients and scaled coupling in the reduced wave number space, so that we must have

$$\eta(r) = e^{\phi(r)}\eta_I(r) \quad \text{and } \zeta(r) = e^{\phi(r)}\zeta_I(r) \tag{40}$$

$$\lambda(r) = e^{3r}\psi(r)\lambda_I(r) \tag{41}$$

and the new force terms are

$$f_i^{S\prime}(\mathbf{k}',\omega') = \frac{e^{\phi(r)}}{\psi(r)}f_i^{S>}(\mathbf{k},\omega) \tag{42}$$

with a similar relation for $f_i^{C\prime}(\mathbf{k}',\omega')$.

Similarly, the condition of RG invariance on the stirring-correlation implies that we must get its form in the reduced range as

$$\langle f_i^{S\prime}(\mathbf{k}_1',\omega_1')f_j^{S\prime}(\mathbf{k}_2',\omega_2')\rangle = \frac{2D(r)}{(k_1')^{4-\epsilon}}P_{ij}(\mathbf{k}_1')(2\pi)^2\delta^2(\mathbf{k}_1'+\mathbf{k}_2') \times(2\pi)\delta(\omega_1'+\omega_2') \tag{43}$$

so that we must have

$$D(r) = \frac{e^{3\phi(r)-(6-\epsilon)r}}{\psi^2(r)}D_I(r). \tag{44}$$

Noting that $D_I(r)=D_0$, and fixing $D(r)$ at the value D_0, we obtain

$$\psi(r) = e^{3\phi(r)/2}e^{-(3-\epsilon/2)r} \tag{45}$$

and hence the scaled coupling is obtained as

$$\lambda(r) = \lambda_0 e^{3\phi(r)/2}e^{\epsilon r/2}. \tag{46}$$

VI. RENORMALIZATION GROUP FLOW

Now we assume that the above elimination-process is repeated in recursive steps of infinitesimal r, so that we may expect to approach a (stable) fixed point after many such infinitesimal steps, i.e. in the UV limit $k\to\infty$. Thus we expand the recursion relations in the limit $r\to 0$. From Eqs. (31), (32), (40), and (46) we get, after assuming the iterative nature of the RG method,

$$\frac{\mathrm{d}\eta}{\mathrm{d}r} = \eta(r)\left[z(r)+\frac{3}{16\pi}\frac{g(r)}{1+\kappa(r)}\right], \tag{47}$$

$$\frac{\mathrm{d}\zeta}{\mathrm{d}r} = \zeta(r)\left[z(r)+\frac{3}{16\pi}\frac{g(r)}{2\kappa(r)}\right], \tag{48}$$

$$\frac{\mathrm{d}\lambda}{\mathrm{d}r} = \frac{1}{2}\lambda(r)[\epsilon+3z(r)], \tag{49}$$

where $z(r)=\mathrm{d}\phi(r)/\mathrm{d}r$, and r is no longer an infinitesimal argument. In the above equations, we have defined the rescaled couplings as

$$g(r) = \frac{\lambda^2(r)D_0}{\eta^3(r)}\Lambda^{\epsilon} \quad \text{and } \kappa(r) = \frac{\zeta(r)}{\eta(r)}, \tag{50}$$

which are similar to the definitions for the corresponding bare quantities, Eqs. (34). Differentiating these quantities with respect to r and using Eqs. (47)–(49) we get the following RG flow equations:

$$\frac{\mathrm{d}g}{\mathrm{d}r} = g(r)\left[\epsilon - \frac{9}{16\pi}\frac{g(r)}{1+\kappa(r)}\right], \tag{51}$$

$$\frac{\mathrm{d}\kappa}{\mathrm{d}r} = \frac{3}{16\pi}\kappa(r)\left[\frac{1}{2\kappa(r)} - \frac{1}{1+\kappa(r)}\right]g(r). \tag{52}$$

A. Case I

For $\epsilon > 0$, it can easily be checked that the scaled couplings $g(r)$ and $\kappa(r)$ approach the nontrivial (UV stable) fixed point (g^*, κ^*) given by

$$g^* = \frac{32\pi}{9}\epsilon \quad \text{and } \kappa^* = 1, \tag{53}$$

the latter being consistent with Kraichnan's result [6].

From Eqs. (40), (46), and (50) we get

$$\eta_I^3(r) = \frac{D_0\Lambda^\epsilon e^{\epsilon r}}{g(r)} \tag{54}$$

from which we can find the renormalized drag $\eta_R(k,0)$ in the RG limit of large k by choosing large r such that $k=\Lambda e^r$, yielding

$$\eta_R(k,0) = \left(\frac{D_0}{g^*}\right)^{1/3} k^{\epsilon/3}. \tag{55}$$

On using this result in the definition for the energy spectrum

$$E(k) = \frac{k}{4\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}Q_R(k,\omega), \tag{56}$$

where the renormalized correlation is given by $Q_R(k,\omega) = |G_R^S(k,\omega)|^2 F^S(k)$ with $G_R^S(k,\omega) = [-i\omega + \eta_R(k,0)]^{-1}$, we get

$$E(k) = \frac{1}{4\pi}(D_0^2 g^*)^{1/3} k^{-3+2\epsilon/3} \tag{57}$$

indicating that $\epsilon=2$ gives the Kolmogorov scaling in the energy regime.

Defining the amplitude-ratio $\mu^2 = \alpha^2/C$ in the energy regime with $\eta(k) = \alpha\bar{\varepsilon}^{1/3}k^{2/3}$ and $E(k) = C\bar{\varepsilon}^{2/3}k^{-5/3}$, we get from Eqs. (55), (57), and (53),

$$\mu^2 = \frac{4\pi}{g^*} = \frac{9}{8\epsilon} = \frac{9}{16} = 0.5626, \tag{58}$$

where we have set the Komogorov value $\epsilon=2$.

To evaluate the Kolmogorov constant, we need one more relation between α and C. Kraichnan numerically integrated the energy transport equation coming from Eq. (12) and obtained $C = 8.94\ \mu^{2/3}$ [6] in two dimensions. Using our RG value of μ^2 in this result yields

$$C = 8.94 \times \left(\frac{9}{16}\right)^{1/3} = 7.3798. \tag{59}$$

The above results may be compared with Kraichnan's exact results [6] $\mu^2 = (0.609)^2 = 0.371$ and $C = 6.69$, obtained via numerical computation of the test-field based closure Eqs. (10)–(12).

B. Case II

Next we consider the case $\epsilon=0$. Substituting this value in the RG flow equation it can be seen that

$$g(r) = \frac{32\pi}{9}\frac{1}{r} \tag{60}$$

for large r. Thus from Eqs. (40), (46), and (50) we obtain

$$\eta_I(r) = \left(\frac{D_0}{g(r)}\right)^{1/3} = \left(\frac{9D_0}{32\pi}r\right)^{1/3} \tag{61}$$

in this case. Choosing large r such that $k=\Lambda e^r$ readily yields

$$\eta_R(k,0) = \left(\frac{9}{32\pi}D_0 \ln\frac{k}{\Lambda}\right)^{1/3}. \tag{62}$$

Use of Eq. (56) then gives

$$E(k) = \frac{1}{4\pi}\left(\frac{32\pi}{9}D_0^2\right)^{1/3} k^{-3}\left(\ln\frac{k}{\Lambda}\right)^{-1/3}. \tag{63}$$

Now we define the amplitude ratio $(\mu')^2 = \beta^2/C'$ in the enstrophy regime with $\eta(k) = \beta\bar{\chi}^{1/3}(\ln k/\Lambda)^{1/3}$ and $E(k) = C'\bar{\chi}^{2/3}k^{-3}(\ln k/\Lambda)^{-1/3}$, so that we get from the above equations

$$(\mu')^2 = \frac{9}{8}. \tag{64}$$

For the evaluation of the Kraichnan-Batchelor constant, we use Kraichnan's analytical result [6] $C' = (\frac{16}{3}\mu')^{2/3}$ coming from the enstrophy transport equation obtained from Eq. (12). This yields

$$C' = \left(\frac{16}{3}\right)^{2/3} \times \left(\frac{9}{8}\right)^{1/3} = 3.1748. \tag{65}$$

The above values are at variance with Kraichnan's analytical results $(\mu')^2 = 9/16$ and $C' = 2.262$ based on the test-field closure Eqs. (10)–(12).

VII. DISCUSSION AND CONCLUSION

In our above calculations, we started with Kraichnan's unclosed test-field dynamics [given by Eqs. (6) and (7)], namely, the passive advection of the solenoidal and compressive parts of a test-field by a purely solenoidal velocity field (with the self-advection terms suppressed in order to estimate the effect of pressure). We added (linear) Rayleigh drag terms in order to model the coupling of the 2D fluid with its 3D environment, as suggested by experiment [17]. Further, we assumed a random driving force field of correlation $\sim k^{-4+\epsilon}$ and carried out an RG iteration by eliminating modes starting with the IR end (unlike the conventional Wilsonian RG scheme). The drag coefficients undergo relevant corrections due to the elimination of scales, and there exists UV attractive fixed points.

We find that the above RG iterations yield the Kolmogorov spectrum $k^{-5/3}$ when $\epsilon=2$ [Eq. (57)] and the Kraichnan-Batchelor spectrum $k^{-3}(\ln k/\Lambda)^{-1/3}$ when $\epsilon=0$ [Eq. (63)]. The respective universal constants were also evaluated by means of evaluating the amplitude ratios μ and μ' in the above RG scheme, yielding $\mu^2=9/16=0.5626$ and $(\mu')^2=9/8$. These values are at variance with Kraichnan's (exact) results $\mu^2=(0.609)^2=0.371$ and $(\mu')^2=9/16$ following from Kraichnan's numerical and analytical calculations based on TFM closure equations (10)–(12).

These departures can be traced when we observe Kraichnan's TFM closure equations (10) and (11) more closely. It may be noted that these integrals involve the quantities $[\eta^{S/C}(k)+\eta^{C/S}(q)+\eta^{S}(p)]$ in the *denominators*. Noting that $\eta^S(k)=\eta^C(k)$ in Kraichnan's 2D calculations, if we expand these integrands in the limit $p\ll k$ (as in our RG calculations), we would obtain

$$\frac{1}{\eta(k)+\eta(q)+\eta(p)} \approx \left(\frac{1}{2}\right)\frac{1}{\eta(k)}.$$

We note that the factor of $\left(\frac{1}{2}\right)$ on the right-hand side is a consequence of three η's in the denominators of Kraichnan's closure Eqs. (10) and (11). These closure equations are the renormalized counterparts of the self-energy integrals [Eqs. (24) and (25)] in our present RG scheme. We can guess that the renormalized self-energy integrals $\Sigma_R^S(k,0)$ and $\Sigma_R^C(k,0)$ corresponding to the RG scheme would have $[\eta_R(q)+\zeta_R(p)]$ and $[\eta_R(q)+\eta_R(p)]$ in the respective denominators. Expanding them in the limit $p\ll k$ gives

$$\frac{1}{\eta_R(q)+\zeta_R(p)} \approx \frac{1}{\eta_R(k)}$$

and the same result for the other integrand. Here we note that there is no factor of $\left(\frac{1}{2}\right)$ on the right-hand side. Thus Kraichnan's closure integrals would give rise to an *extra* factor of $\left(\frac{1}{2}\right)$ as a consequence of three η's, as seen above. When we take this extra factor of $\left(\frac{1}{2}\right)$ into consideration, we expect that $(\mu')^2_{\mathrm{RG}}=2(\mu')^2_{\mathrm{TFM}}$. Indeed we see that this is true; Kraichnan obtained $(\mu')^2_{\mathrm{TFM}}=9/16$, whereas our RG calculations give $(\mu')^2_{\mathrm{RG}}=9/8$. Thus we see that the difference between Kraichnan's TFM and the present RG solely results from the fact that TFM has three η's in the denominator in the response equation whereas the RG scheme generates only two renormalized drag coefficients in the self-energy integrals.

Similar results can be obtained by extending similar arguments in the energy regime as well. Having three η's in Kaichnan's closure while only two η's in RG in the respective denominators, it is straightforward to guess that $\mu^2_{\mathrm{TFM}}<\mu^2_{\mathrm{RG}}$. We can obtain another bound by comparing the $p\ll k$ expansion of three η's, namely

$$\frac{1}{k^{2/3}+q^{2/3}+p^{2/3}} = \left(\frac{1}{2}\right)\frac{1}{k^{2/3}}\left[1+\frac{\mathbf{k}\cdot\mathbf{p}}{3k^2}-\frac{1}{2}\left(\frac{p}{k}\right)^{2/3}+\cdots\right],$$

with that of two η's, namely

$$\frac{1}{q^{2/3}+p^{2/3}} = \frac{1}{k^{2/3}}\left[1+\frac{2\mathbf{k}\cdot\mathbf{p}}{3k^2}-\left(\frac{p}{k}\right)^{2/3}+\cdots\right].$$

The $\mathbf{k}\cdot\mathbf{p}$ terms contribute zero on integration. The $(p/k)^{2/3}$ terms differ by a factor of $\frac{1}{2}$. This comparison indicates that $\mu^2_{\mathrm{TFM}}>\left(\frac{1}{2}\right)\mu^2_{\mathrm{RG}}$. Thus $\mu^2_{\mathrm{TFM}}<\mu^2_{\mathrm{RG}}<2\mu^2_{\mathrm{TFM}}$. Indeed we see in the energy regime that the RG estimate $\mu^2_{\mathrm{RG}}=0.5626$ is about 1.5 times higher than Kraichnan's numerical result $\mu^2_{\mathrm{TFM}}=0.371$. Thus we see again that the difference in the two results (TFM and RG) arises due to different number of η's in the denominators.

We would like to note that we are unable to apply a similar approach for the comparison of Yakhot and Orszag's [32] Renormalization Group (YO-RG) results in 3D turbulence. As Kraichnan's Eulerian DIA (or any other Eulerian closure based on the full NSE) fails to represent the Kolmogorov regime, it is thus unable to yield any result related to the Kolmogorov cascade. On the other hand, the randomly forced models of Forster, Nelson, and Stephen [30] and Yakhot and Orszag [32] based on the full NSE have been successful by means of using the Wilsonian RG scheme. However, Eyink [49] pointed out that the random forcing RG model requires an infinite number of loops for an $O(y)$ calculation (see also [43,44], here y is defined according to [46]). This suggests the invalidity of the one-loop RG calculations relying on the lowest-order $O(y)$ expansion. However, Orszag and Yakhot [50], taking the case of a passive scalar, presented a detailed analysis by carrying out an exact summation of all higher order diagrams within a Wilson-type summation scheme, yielding amplitudes differing by only a few percent from the lowest order result and gave results close to the fixed point value, thus giving confidence in the RG sceme. It is also interesting to note that a different type of randomly forced RG has been carried out in Ref. [51].

The most interesting feature of our present RG calculations is that we introduced linear drag terms (the Rayleigh drag terms) in the dynamical equations in order to model the drag operating at large scales because of frictional coupling of the 2D fluid with its 3D environment. This was suggested by the experiment of Rivera and Wu [17] which indicated the validity of the linear drag model in the turbulence in a soap film. We find in our calculations that the Rayleigh drag coefficients η_0 and ζ_0 do indeed acquire *relevant* corrections, suggesting that the dynamical nonlinearities generating the turbulent cascades are "in tune" with such large scale drag mechanism. Thus our RG calculation supports this picture of *linear* (or Rayleigh) drag model.

We close our discussion by noting a few recent results from numerical experiments. Boffeta *et al.* [52] carried out a high resolution numerical simulation of the inverse energy cascade regime. They found no measurable corrections due to intermittency and thus confirmed the Kolmogorov scaling $k^{-5/3}$ in two dimensions. A particular ingradient in their modeling was the introduction of a *linear* drag term with a drag coefficient α chosen so as to make the friction scale $\ell_f \sim (\bar{\varepsilon}/\alpha^3)^{1/2}$ much smaller than the system size. This ensured the absence of finite size effects, namely the Bose-Einstein condensation [15], the presence of which would otherwise tend to destroy the Kolmogorov regime. Another simulation, for the direct enstrophy cascade regime, was carried out by Pasquero and Falkovich [53]. In order to avoid the formation of coherent structures, the ingredients in their simulation were the choice of a thin-band forcing with random phases together with a linear drag term and an eighth-order hyperviscosity term for fine-scale viscous draining. Their simulation supported the Kraichnan-Batchelor spectrum [Eq. (2)] along with the logarithmic renormalization.

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